ON THE SPACE REFLECTIONS DEFINITION PROBLEM IN THE MAGNETIC CHARGE THEORY

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Abstract

A new way to define the operation of P-inversion in the theory with a magnetic charge is presented.

At this conference, dedicated to memory of Professor A.O.Barut, it is seemed to be pertinent to consider in a rather retrspective view some fragments of his rich scientific legacy. Ecseptionally active scientific and human approachability by Professor Barut was his modus vivendi. He always was open for collaboration.

The P-invariant quantum mechanical dichotomic model of charge-monopole interaction [1] proposed by Professor Barut more than twenty years ago is demonstrated below as an example his original ideas influence upon our investigations. Later this influence was developed in a very fruitful direct scientific cooperation and in the same time in imforgetable personal contacts.

The problem of P invariance of the electrodynamics with electric and magnetic charges has a very long story (see, e.g.[2] - [4]). Despite the quite

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definite answer has been obtained in [5, 6] there still are some new endeavours to get a progress in this direction. The gist of the matter can be easily be demonstrated on the example of classical field theory.

Let us consider the pure SU(2) gauge model with the Lagrangian

$$L = -\frac{1}{4} F^a_{\mu,\nu} F^{a\mu,\nu}, \tag{1}$$

where the field tensor is

$$F^a_{\mu,\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - e\varepsilon_{abc} A^b_\mu A^c_\nu, \qquad (a = 1, 2, 3), \tag{2}$$

and A^a_μ is the gauge potential.

As is well known the corresponding equations of motion have the Wu-Yang solution

$$A^a_k = \varepsilon_{akm} \frac{r_m}{r^2}. \tag{3}$$

It is obvious that the model is invariant under the reflection $\mathbf{r} \to -\mathbf{r}$ and the Wu-Yang potential is transformed as a vector: $A^a_k(\mathbf{r}) \to -A^a_k(\mathbf{r})$. But on the other hand there is an abelization of the solution (3) by means of the gauge transformation

$$SA^a_k(\mathbf{r})S^{-1} + \frac{i}{e}(\nabla_k S)S^{-1} = A^D_k(\mathbf{r}, \mathbf{n})\sigma_3, \tag{4}$$

where

$$S = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} e^{-i\phi} \\ \sin\frac{\theta}{2} e^{i\phi} & \cos\frac{\theta}{2} \end{pmatrix}, \tag{5}$$

and

$$A^D_a(\mathbf{r}, \mathbf{n}) = \frac{\varepsilon_{abc} r_b n_c}{r - (r_a n_a)}$$

is the Dirac potential.

Of course, the gauge transformation (4) can not violate the P invariance of the theory. This point gives a possibility to obtain a new way to define the operation of P inversion in the theory with a magnetic charge.

Indeed, under the P-inversion $\theta \to \pi - \theta, \quad \phi \to \phi + \pi$ equation (4) is transformed as

$$S_p(-A^a_k)S_p^{-1} - \frac{i}{e}(\nabla_k S_p)S_p^{-1} = A^D_k(-\mathbf{r}, \mathbf{n})\sigma_3,$$

or

$$S_p A^a_k S_p^{-1} + \frac{i}{e}(\nabla_k S_p)S_p^{-1} = -A^D_k(-\mathbf{r}, \mathbf{n})\sigma_3, \tag{6}$$

where

$$S_p = \left(\begin{array}{cc} \sin\frac{\theta}{2} & \cos\frac{\theta}{2}e^{-i\phi} \\ -\cos\frac{\theta}{2}e^{i\phi} & \sin\frac{\theta}{2} \end{array} \right). \tag{7}$$

We have to transform Eq.(6) to the (4) if we like to check the P invariance of the model. To this end we write

$$S_p = RS, \quad \text{i.e.} \quad R = S_p S^{-1} = \begin{pmatrix} 0 & e^{-i\phi} \\ -e^{i\phi} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}$$

Then, bring the matrix of transformation R from the left to right in Eq.(6) we really get the Eq.(4) because the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ changes the common sigh of the r.h.s. and the matrix $\begin{pmatrix} -e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix}$ changes the sigh of $\mathbf{r}$. Thus the r.h.s. of (4) is a true vector up to a gauge transformation determined by R^{-1}. Consequently the Dirac potential $\mathbf{A}^D(\mathbf{r}, \mathbf{n})$ is a pseudovector up to $U(1)$ gauge transformation determined by $\exp(i\phi)$. The central point of our cosideration is transformational properties of matrix S under spatial reflection. Let $\mathbf{A}_k^a(\mathbf{r})$ in eq. (4) be any true vector, then the corresponding r.h.s. is a vector up to the gauge transformation determined by $S\,S_\gamma^{-1}$. It is worth noting that in quantum theory the use of the unitary transformations determined by (5) leads also to redefinition of the P-inversion operator

$$P' = SPS^\dagger = S(S_p)^{-1}P = RP. \tag{8}$$

Obviously, P' is a Hermitian operator, i.e.

$$(P')^\dagger = P(R^{-1})^\dagger = R^{-1}P.$$

The matrix R up to nonsignificant here factor σ_3 concided with the matrix

$$\hat{R} = \left(\begin{array}{cc} 0 & e^{-2i\mu\phi} \\ -e^{2i\mu\phi} & 0 \end{array} \right), \qquad \mu = eg = 1/2, \tag{9}$$

introduced in [7]. It is easy to check that using the combination of the standard reflection of the space coordinates and the gauge transformation (9) allows to restore formal P invariance of the model with Barut's dichotomic Hamiltonian [1], [9]

$$H = H(\mathbf{A}^D \sigma_3)$$

But we seems to run into a slight problem if we attempt to include other external electromagnetic fields $\mathbf{A}^e$ into this framework together with $\mathbf{A}^D$.

Indeed, if we are to follow the usual embedding procedure all the electromagnetic potentials have to be multiplied by σ_3. Only in this case is this model effectively equivalent to the U(1) model with the standard rule of adding fields. In this case the non-relativistic Hamiltonian operator of the charge-dyon system would be

$$H = -\frac{1}{2M}(\mathbf{P} + (e\mathbf{A}^D + e\mathbf{A}^e)\sigma_3)^2 - eA_0^D\sigma_3 - eA_0^e\sigma_3. \qquad (10)$$

Instead we can write a P-invariant Hamiltonian operator as

$$H = -\frac{1}{2M}(\mathbf{P} + e\mathbf{A}^D\sigma_3 + e\mathbf{A}^e I)^2 - eA_0^D\sigma_3 - eA_0^e I. \qquad (11)$$

This, as a matter of fact, implies an extension of electrodynamics because the model with Hamiltonian (11) is invariant under the gauge transformation with $U(1) \otimes U(1)$ group and describes the interaction of the quantum particle carrying pseudoscalar ($e\sigma_3$) and scalar (eI) charges with pseudovestor $(A_0^D, \mathbf{A}^D)$ and vector $(A_0^e, \mathbf{A}^e)$ fields, correspondingly. Moreover, with this definition it is not necessary to fix the same interaction constant e. Indeed, because the eigenfunction Ψ of the Hamiltonian operator (11) is a two-component entity, the natural gauge transformation is $\Psi \to \exp\{i(e + e'\sigma_3)\}\Psi$.

Let us illustrate these arguments in the example ot the calculation of selection rules for dipole radiation.

It is easy to see that the dipole moment operator corresponding to the Hamiltonian (10) is $e(\mathbf{r}\sigma_3)$ and the corresponding operator for the Hamiltonian (11) s $e(\mathbf{r}I)$. In the first case the total set includes the charge operator $e\sigma_3$, therefore a matrix element should be calculated between the following type of wavefunctions:

$$\begin{pmatrix} \Phi_{Njm\mu}(\mathbf{r}) \\ 0 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} 0 \\ \Phi_{Njm\mu}(\mathbf{r}) \end{pmatrix},$$

where $\mu = eg$, $\Phi_{Njm\mu}(\mathbf{r}) = R_{Nj}(r)Y_{jmq}(\theta, \varphi)$ are the wavefunctions of the electrically charged particles in the field $\mathbf{A}^D$, $R_{Nj}(r)$ is their radial part, j and m are the quantum numbers corresponding to the eigenvalues of the operator $\mathbf{J}^2$ and J_3 ($\mathbf{J}$ is the operator for the total angular momentum of the system) and the generalized spherical harmonics $Y_{jmq}(\theta, \varphi)$ are expressed through the standard Jacobu polynomial $P_n^{(\alpha,\beta)}(x)$[10]:

$$Y_{jmq}(\theta, \varphi) = N(1-x)^{-(m+\mu)/2}(1+x)^{-(m-\mu)/2}P_{j+m}^{(-m-\mu),(-m+\mu)}(x)e^{i(m+\mu)}\varphi, \qquad (12)$$

where $x = \cos\theta$, and

$$N = 2^m \sqrt{\frac{(2j+1)(j-m)!(j+m)!}{4\pi(j-\mu)!(j+\mu)!}}.$$

After integration

$$\int d^3x \Phi^*_{Njm\mu}(\mathbf{r})\mathbf{r}\sigma_3\Phi_{Njm\mu}(\mathbf{r}), \tag{13}$$

we obtain the selection rules that coincide with results of [5]:

$$\Delta j = 0, \pm 1, \quad \Delta m = 0, \pm 1 \tag{14}$$

Thus, the parity violating transitions with $\Delta j = 0$ are allowed together with the standard transitions with $\Delta j = \pm 1$.

In the second case however, among operators commuting with Hamiltonian (11) there are the operator σ_3 and generalized P inversion operator (8), which however do not commute with each other. Choosing the general eigenfunctions of the operators (8) and (11) as

$$\Psi_{Njm\mu}(\mathbf{r}) = \begin{pmatrix} \Phi_{Njm\mu}(\mathbf{r}) \\ \Phi_{Njm-\mu}(\mathbf{r}) \end{pmatrix}, \tag{15}$$

we calculate the matrix element of the dipole moment operator

$$\int d^3x \Psi^*_{Njm\mu}(\mathbf{r})\mathbf{r}I\Psi_{Njm\mu}(\mathbf{r}), \tag{16}$$

and integrate over the angular part of (16). We have for instance:

$$\begin{aligned}
&\int_\Omega d\Omega \Psi^*_{j'm'\mu'}(\theta,\varphi)\cos\theta I \Psi_{jm\mu}(\theta,\varphi) = \\
&C\Bigg(\int_\Omega d\Omega Y_{j'm'\mu'}(\theta,\varphi)Y_{100}(\theta,\varphi)Y_{jm\mu}(\theta,\varphi) \\
&\pm \int_\Omega d\Omega Y_{j'm'\mu'}(\theta,\varphi)Y_{100}(\theta,\varphi)Y_{jm-\mu}(\theta,\varphi)\Bigg) \\
&= C'\begin{pmatrix} j' & 1 & j \\ -m & 0 & m \end{pmatrix}\left[\begin{pmatrix} j' & 1 & j \\ -\mu & 0 & \mu \end{pmatrix} - \begin{pmatrix} j' & 1 & j \\ \mu & 0 & -\mu \end{pmatrix}\right],
\end{aligned} \tag{17}$$

where C and C' are some constants. Taking into account the familiar properties of the $3-j$ symbols

$$\begin{pmatrix} j' & 1 & j \\ -\mu & 0 & \mu \end{pmatrix} = (-1)^{j'+j+1}\begin{pmatrix} j' & 1 & j \\ \mu & 0 & -\mu \end{pmatrix}$$

we see that the integral (17) is non-zero only for $\Delta j = \pm 1$. In an analogous way, after integration of the angular integrals of $\sin\theta e^{\pm i\varphi}$ we obtain the selection rules:

$$\Delta j = \pm 1, \quad \Delta m = 0, \pm 1. \tag{18}$$

that is to say the dipole transitions with parity violation are absent in this case.

Let us note that the use of the wavefunction (15) in (13) does not modify the selection rules (14) but the value of the integral (13) in this case is increased twofold.

Thus the restoring of the standard selection rules and P invariant description of this system are achieved by means of the gauge group extension to $U(1)\otimes U(1)$.At the same time the hypothesis about the Abelian magnetic charge is not connected with an extension of the symmetry group but based on the transition to the non-trivial fiber-bundle over the space-time base with the structure group $U(1)$ when the connection (potential) and sections (wavefunctions) cannot be described globally.

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